REAL ANALYSIS WITH TOPOLOGY
TOPIC II - NATURAL NUMBERS

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Abstract. The document reviews the main properties of the natural numbers, including well-ordering and induction.

1. The Well-Ordering Principle

The set of *natural numbers* is $\mathbb{N} = \{0, 1, 2, 3, \dots\}$, as characterized by the five *Peano axioms*. These use the idea of *successor*: the successor of n is $n + 1$.

Axiom 1. (Peano's Axioms) *The set* $\mathbb{N}$ *of natural numbers satisfies*

- **(N1)** *1 belongs to* $\mathbb{N}$
- **(N2)** *If* n *belongs to* $\mathbb{N}$*, then its successor belongs to* $\mathbb{N}$
- **(N3)** *1 is not the successor of any element in* $\mathbb{N}$
- **(N4)** *If* n *and* m *in* $\mathbb{N}$ *have the same successor, then* $n = m$
- **(N5)** *A subset of* $\mathbb{N}$ *which contains 1, and which contains the successor of each of its elements, must equal* $\mathbb{N}$.

The main axiom here is **(N1)**, which we may call the *induction axiom*. We restate this as

Proposition 1. (Peano's Rule)
Let $S \subset \mathbb{N}$*. If*

- **(a)** $1 \in S$*, and*
- **(b)** $n \in S \Rightarrow n + 1 \in S$,

then $S = \mathbb{N}$.

Proposition 2. (Well-Ordering Principle)
Let $X \subset \mathbb{N}$ *be nonempty. Then there exists* $a \in X$ *such that* $a \leq x$ *for every* $x \in X$.

Proof. Let $X \subset \mathbb{N}$ and assume that X has no smallest element; we show that $X = \varnothing$. Let

$$S = \{n \in \mathbb{N} \mid n < x \text{ for every } x \in X\}.$$

Clearly $S \cap X = \varnothing$; if we show that $S = \mathbb{N}$, then $X = \varnothing$.

Since 1 is less than or equal to every natural number, 1 is less than or equal to every natural number in X. Since X has no smallest element, $1 \notin X$, so $1 < x$ for every $x \in X$. Thus $1 \in S$.

Suppose that $n \in S$. Then $n < x$ for every $x \in X$, so $n + 1 \leq x$ for every $x \in X$. If $n + 1$ were in X, it would be the smallest element of X; since X has no smallest element, $n + 1 \notin X$; thus $n + 1 \neq x$ for every $x \in X$, whence $n + 1 < x$ for every $x \in X$. It follows that $n + 1 \in S$, and by Peano's Rule, $S = \mathbb{N}$. □

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2. The Induction Principles

Proposition 3. (Induction Principle)
Let $\{p_i \mid i \in \mathbb{N}\}$ be a set of propositions indexed by $\mathbb{N}$. Suppose that

(I1) *p_1 is true;*
(I2) *p_{n-1} implies p_n, for $n > 1$.*

Then p_i is true for all $i \in \mathbb{N}$.

Proof. Suppose not, and let $n \in \mathbb{N}$ be the smallest natural number such that p_n is false. Then $n \neq 1$, since p_1 is true by **(I1)**, so $n - 1$ exists as a natural number. Since $n - 1 < n$, p_{n-1} is true. By **(I2)**, $p_{n-1} \Rightarrow p_n$, so p_n is true, contradicting the assumption. Thus p_i is true for all $i \in \mathbb{N}$. □

We call **(I1)** the *base case* and **(I2)** the *inductive step.* We note that by shifting, we can actually start the induction at any integer. Here is an example demonstrating proof by induction.

Example 1. Show that $11^n - 4^n$ is a multiple of 7 for all $n \in \mathbb{N}$.

Proof. A natural number a is a multiple of 7 if and only if $a = 7b$ for some natural number b. We proceed by induction on n. First we verify the base case, when $n = 1$, and then demonstrate the induction step, wherein we show that if the proposition is true for $n - 1$, then it is true for n.

(I1) Let $n = 1$. Then $11^1 - 4^1 = 7 = 7 \cdot 1$, which is a multiple of 7, so $11^n - 4^n$ is a multiple of 7 in this case. This verifies the base case.

(I2) Let $n > 1$, and assume that $11^{n-1} - 4^{n-1}$ is a multiple of 7. Then $11^{n-1} - 4^{n-1} = 7k$ for some $k \in \mathbb{N}$. Now compute

$$\begin{aligned} 11^n - 4^n &= 11^n - 11 \cdot 4^{n-1} + 11 \cdot 4^{n-1} - 4^n \\ &= 11(11^{n-1} - 4^{n-1}) + 4^{n-1}(11 - 4) \\ &= 11 \cdot 7k + 4^{n-1} \cdot 7 \\ &= 7(11k + 4^{n-1}), \end{aligned}$$

which is a multiple of seven.

Thus properties **(I1)** and **(I2)** hold, so the proposition is true for all $n \in \mathbb{N}$. □

Proposition 4. (Strong Induction Principle)
Let $\{p_i \mid i \in \mathbb{N}\}$ be a set of propositions indexed by $\mathbb{N}$. Suppose that

(IS) *if p_i is true for all $i < n$, then p_n is true.*

Then p_i is true for all $i \in \mathbb{N}$.

Proof. Suppose not, and let m be the smallest natural number such that p_m is false. Then p_i is true for all $i < m$. By **(IS)**, p_m is true, contradicting the assumption. Thus p_i is true for all $i \in \mathbb{N}$. □

It is common in the statement of the strong induction principle to include the base case **(I1)**, that p_1 is true, as a premise. In practice, we may have to verify **(I1)** as a step in demonstrating **(IS)**. We note that **(I1)** is implied by **(IS)**, but that **(I2)** is not implied by **(IS)** (why?).

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